EVENTUALLY AREALLY MEAN p-VALENT FUNCTIONS

BY

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ABSTRACT

Theorems concerning areally mean p-valent functions are extended to eventually areally mean p-valent functions. In particular, suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is eventually areally mean p-valent in the unit disc, b, c are positive integers, $a \geqq \max\{p-1, 0\}$. If $|a_n| \leqq Cn^{\alpha}$ for all $n = bm + c$, $m = 1, 2, \ldots$, then $|a_n| \leqq C' n^{\alpha}$ for all n. This is a marked extension of results due to Goluzin and to Hayman.

1. Introduction

Areally mean p-valent functions (hereafter abbreviated as *ampv*) are known to possess many of the same coefficient and growth properties of p-valent functions. At the same time *ampv* functions have the disadvantage of being unduly sensitive to zeros since they may have no more than $[p]$ zeros where $[p]$ denotes the integer part of p.

To remove this unnecessary sensitivity to zeros and yet retain the growth properties and coefficient estimates of *ampv* functions we introduce and investigate the class of eventually areally mean p-valent functions (*eampv*). Since an *ampv* function may be eventually areally mean q-valent (*eamqv*) with q considerably smaller than p, this theory makes possible even more precise information about *ampv* functions in certain situations. In addition the theory provides a means of obtaining knowledge concerning functions which are not *amqv* for any q.

In particular, we show that if $f(z)$ is *eampv* then $M(r,f) = O(1-r)^{-2p}$ and the set of points where the order of f is positive is countable and satisfies $\sum \alpha(\zeta) \leqq 2p$. If $f(z)$ is *eampv* and equal to

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$$\sum_{j=0}^{N-1} a_j z^j + \sum_{j=0}^{\infty} a_{N+jk} z^{N+jk},$$

then $M(r,f) = O(1-r)^{-2p/k}$. Furthermore, if $f(z) \in U_\beta$ is *eampv* then $M(r,f) = O(1-r)^{-\alpha}$ implies $|a_n| = O(n^{\alpha-1})$ for all $\alpha > 3/2$. Other interesting coefficient results are obtained by placing various controls on the rate of growth of the area of $f(|z| < r)$ over a fixed disc centered at the origin in the image plane.

2. Preliminary remarks

If $f(z)$ is analytic in the unit disc $D = \{|z| < 1\}$, let $n(w)$ be the number of roots in D of the equation $f(z) = w$, counted according to their multiplicity. As is standard notation we let

$$p(R) = p(R, D, f) = \frac{1}{2\pi} \int_0^{2\pi} n(R \exp(i\theta)) d\theta \tag{2.1}$$

$$h(R) = p(R) - p \tag{2.2}$$

$$W(R, R_0) = \int_{R_0}^{R} p(\rho) d(\rho^2) = pR^2 + H(R, R_0) = H^*(R) \tag{2.3}$$

$$W(R,0) = W(R) = pR^2 + H(R). \tag{2.4}$$

A function $f(z)$ analytic in D is said to be

(i) *p-valent* if and only if $n(w) \leqq p$, for all $w \in \mathbb{C}$,

(ii) *eventually p-valent* if and only if there is an R_0 such that $n(w) \leqq p$ for all $|w| \geqq R_0$,

(iii) *areally mean p-valent* if and only if $W(R) \leqq pR^2$ for all $R \geqq 0$,

(iv) *(Spencer) eventually areally mean p-valent* if and only if there is an R_0 such that $W(R) \leqq pR^2$ for all $R \geqq R_0$,

(v) *eventually areally mean p-valent* if and only if there is an R_0 such that $W(R, R_0) \leqq pR^2$ for all $R \geqq 0$.

Note that an *eampv* function with $R_0 = 0$ is *ampv* and any *ampv* function is *eampv* with respect to any $R_0 \geqq 0$. After the completion of the original work in this paper, which was based on [6], my attention was directed to Spencer's original papers [10], [11]. One sees immediately that Spencer's eventually areally mean p-valent functions are *eampv*. The converse is drastically false since *eampv* functions may have $W(R_0) = \infty$ while Spencer *eampv* functions necessarily have $W(R_0) < \infty$.

A surface S over $\mathbb{C}$ is said to have locally finite area at a point $w \in \mathbb{C}$ if there is some neighborhood N of w such that the area of that part of the surface S which lies over N is finite. Thus while the image of *ampv* and Spencer *eampv* functions must be of locally finite area *everywhere* in the image domain, the image of *eampv* functions can fail to be of locally finite area everywhere inside the arbitrarily large compacta $|w| \leqq R_0$.

Actually it appears that the notion of Spencer *eampv* functions is due to Littlewood [7]. At the conclusion of Spencer's first paper (submitted in 1938 but not published till 1941) Littlewood remarks that, due to the problems of communication during the war, he not only had to referee the paper but also supply corrections himself. Hence he has added to Spencer's hypothesis the restriction that $p \geqq 1$ (originally stated for $p > 0$) although he remarks that this could be removed if an additional constant is included and $W(R) \leqq pR^2$ holds for $R \geqq R_0$. Spencer, apparently, was not convinced of Littlewood's addition, for in his sequel [11] (published after he had seen Littlewood's remark) he asserts that some of the results in his first paper (Th. 1 in particular) "require the full strength of $W(R) \leqq pR^2$ for all $R > 0$". He does maintain, however, that the hypothesis $W(R) \leqq pR^2$ for $R \geqq R_0 > 0$ is sufficient for the results of his second paper [11]. In any case, we will find that the proof can be generalized and simplified to the case of *eampv* functions.

3. Eventually areally mean p-valent functions

The proofs presented in this paper follow very closely those given by Hayman [6] for *ampv* functions. This is quite surprising when one notes how heavily and how often Hayman's proofs rely on the fact that *ampv* functions have only a finite number of zeros and therefore, for δ sufficiently small, there are no zeros in the annulus $1 - \delta < |z| < 1$. In order that the results of this paper may be referred to in future papers that generalize some of Eke's work [4] on *ampv* functions and to emphasize the parallel with Hayman's proofs, I have adopted Hayman's notation and cite his theorems by number as they appear in [6].

We begin with the observation that if $f(z)$ is *eampv* with respect to R_0, then for all $R \geqq R_0$

$$0 \geqq H^*(R) \geqq -pR^2. \tag{3.1}$$

Consequently,

LEMMA 1. ([6, Lem. 2.1].) *If $f(z)$ is eampv in D with respect to R_0 and $R_1, R_2 \geqq R_0$, then*

$$\int_{R_1}^{R_2} \frac{d\rho}{\rho p(\rho)} \geqq \frac{1}{p} \left\{ \log \frac{R_2}{R_1} - \frac{1}{2} \right\}.$$

PROOF. Exactly as in [6], we have

$$\int_{R_1}^{R_2} \frac{d\rho}{\rho p(\rho)} \geqq \frac{1}{p} \left\{ \log \frac{R_2}{R_1} - \frac{1}{p} \frac{H^*(R_2)}{2R_2^2} + \frac{1}{p} \frac{H^*(R_1)}{2R_1^2} - \frac{1}{p} \int_{R_1}^{R_2} \frac{H^*(\rho)d\rho}{\rho^3} \right\}.$$

An application of (3.1) concludes the proof.

The following theorem is a basic tool for *eampv* functions. Its proof is extremely simple in comparison to Hayman's proof, especially when one notes that we have removed all restrictions on the number of zeros of $f(z)$.

THEOREM 2. ([6, Th. 2.4].) *Let $f(z)$ be analytic in D. Let*

$$\rho_0 = \max\{|f(z)| : |z| = \tanh \tfrac{1}{4}\pi\}.$$

If $\rho_1 = \max\{|f(z)| : |z| = r\}$ *where r satisfies* $\tanh \frac{1}{4}\pi < r < 1$, *then for any* $\rho \geqq \rho_0$

$$\int_{\rho}^{\rho_1} \frac{d\rho}{\rho p(\rho)} < 2 \log \frac{1+r}{1-r} + 2\pi.$$

PROOF. Without loss of generality we may assume that

$$\rho_1 = \max\{|f(z)| : |z| = r\}$$

is taken on at $z = r$, $\tanh \frac{1}{4}\pi < r < 1$. Let $\zeta = \log[(1+z)/(1-z)] = \sigma + i\eta$ and set $g(\zeta) = f[(e^\zeta - 1)/(e^\zeta + 1)]$. Let $\sigma_1 = \log[(1+r)/(1-r)]$. Then $|g(\sigma_1)| = |f(r)| = M(r,f) = \rho_1$.

Since $|g(\frac{1}{2}\pi)| = |f(\tanh \frac{1}{4}\pi)| \leqq \rho_0$ we may therefore conclude that for any $\zeta = \sigma + 0i$, $\frac{1}{2}\pi \leqq \sigma \leqq \sigma_1$, the function $|g(\zeta)|$ takes on each value in $[\rho_0, \rho_1]$ at least once. For each $\rho \in [\rho_0, \rho_1]$ let σ be the first real number $\geqq \frac{1}{2}\pi$ such that $|g(\sigma)| = \rho$. Consequently, $0 < \frac{1}{2}\pi \leqq \sigma \leqq \sigma_1$ and for each $0 \leqq t < \sigma$ we have

$$|g(t)| \leqq \max\{|g(t)| : 0 \leqq t \leqq \tfrac{1}{2}\pi, \tfrac{1}{2}\pi \leqq t < \sigma\} \leqq \max\{\rho_0, \rho\} = \rho.$$

Let $R = \{\zeta = \sigma + i\eta : -\frac{1}{2}\pi < \sigma < \frac{1}{2}\pi + \sigma_1, |\eta| < \frac{1}{2}\pi\}$. For each $\rho \in [\rho_0, \rho_1]$ let C_ρ be the level curve of $|g(\zeta)| = \rho$ which goes through $\sigma \in R$ on which $|g(\zeta)| = \rho$. If we omit the countable number of ρ on which the level curve goes through a zero of $g'(\zeta)$, then we see that we can extend C_ρ either to the boundary of R in both directions or until C_ρ intersects itself. In the first case we have $l(\rho)$, the length of C_ρ, $\geqq \pi$. In the second case, because $g'(\zeta) \neq 0$ on C_ρ, we know that C_ρ is an analytic Jordan curve. In the interior of C_ρ we have $|g(\zeta)| < \rho$ by the maximum principle. However $|g(\zeta)| > \rho$ in a neighborhood of the exterior of C_ρ. Consequently, since $|g(\zeta)| < \rho$ on $[0, \sigma]$, we must have this segment in the interior of C_ρ. Therefore the length of C_ρ must be $\geqq 2\sigma \geqq \pi$. Therefore, for all but a countable set of ρ's in $[\rho_0, \rho_1]$, we have $l(\rho) \geqq \pi$. By Ahlfor's Length Area theorem [6, p. 18]

$$\pi^2 \int_{\rho_0}^{\rho_1} \frac{d\rho}{\rho p(\rho)} \leqq \int_{\rho_0}^{\rho_1} \frac{l^2(\rho) d\rho}{\rho p(\rho)} \leqq 2\pi \cdot \pi(\sigma_1 + \pi)$$

and therefore

$$\int_{\rho_0}^{\rho_1} \frac{d\rho}{\rho p(\rho)} \leqq 2\left(\log\left(\frac{1+r}{1-r}\right) + \pi\right)$$

from which (3.3) is obvious. (Compare to [8, Th. 1.4].)

Our first application of Th. 2 is to generalize a theorem of Cartwright and Spencer for *ampv* functions.

THEOREM 3. ([6, Th. 2.5].) *If $f(z)$ is eampv with respect to R_0, then*

$$M(r,f) \leqq A(p,f,R_0)(1-r)^{-2p} \text{ for } 0 \leqq r < 1$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$.

PROOF. From Lemma 1 and Theorem 2 we have for $R_2 = M(r,f)$, $R = M(\tanh \frac{1}{4}\pi, f)$, $R_1 = \max(R_0, R)$ that

$$\frac{1}{p}\left\{\log\frac{R_2}{R_1} - \frac{1}{2}\right\} \leqq \int_{R_1}^{R_2} \frac{d\rho}{\rho p(\rho)} < 2\log\frac{1+r}{1-r} + 2\pi;$$

hence, $M(r,f) \leqq R_1 2^{2p} \exp(2\pi - \frac{1}{2}) \cdot (1-r)^{-2p}$ for $\tanh \frac{1}{4}\pi < r < 1$. Consequently $M(r,f) \leqq A(f, R_0, p)(1-r)^{-2p}$ for $0 < r < 1$.

Since the class is not a normal family we cannot hope to find a constant in Theorem 3 which is independent of the function. Note that, for fixed p, the family of all locally univalent eventually p-valent functions of the form $f(z) = z + \cdots$ (which are analytic in D) does not form a normal family [2, Cor. 1].

The following is a key theorem for future papers on *eampv* functions (just as its Hayman parallel is key for so many *ampv* theorems).

THEOREM 4. ([6, Th. 2.6].) *Let $f(z)$ be eampv in D with respect to R_0. Let the k discs $|z - z_n| < r_n$ (for $1 \leqq n \leqq k$) be nonoverlapping subsets of D. Let*

$$R_1 = \max\left\{R_0,\ \max_{1 \leqq n \leqq k} \max\{|f(z_n + r_n\zeta)| : |\zeta| \leqq \tanh \tfrac{1}{4}\pi\}\right\}.$$

If $|f(z_n')| \geqq R_2 > eR_1$ *and* $\delta_n = (r_n - |z_n' - z_n|)/r_n > 0$, *then*

$$\sum_{n=1}^{k} [\log(2e^{\pi}/\delta_n]^{-1} < 2p/[\log(R_2/R_1) - 1]. \tag{3.4}$$

PROOF. Although this proof follows the form of Hayman, we will prove the theorem in some detail to facilitate the reader's task and to permit abbreviation of the general procedure in the following proofs of Theorems 5, 6 and 7.

Let D_n be the disc $|z - z_n| < r_n$ and let $p_n(r) = p(R, D_n, f)$. Consider $\phi(\zeta) = f(z_n + r_n\zeta)$. Then $p_n(R) = p(R, |\zeta| < 1,\ \phi(\zeta))$. We choose $\zeta_n = (z_n' - z_n)/r_n$. Then $|\phi(\zeta_n)| \geqq R_2$ and by Theorem 2 we have

$$\int_{R_1}^{R_2} \frac{d\rho}{\rho p_n(\rho)} \leqq \int_{R_1}^{M(|\zeta_n|,\phi)} \frac{d\rho}{\rho p_n(\rho)} < 2\log\frac{1 + |\zeta_n|}{1 - |\zeta_n|} + 2\pi,$$

or

$$\int_{R_1}^{R_2} \frac{d\rho}{\rho p_n(\rho)} \leqq 2\log\left(\frac{2e^{\pi}}{\delta_n}\right).$$

Using Schwarz inequality, we obtain

$$\sum_{n=1}^{k}\left[\log\frac{2e^{\pi}}{\delta_n}\right]^{-1} \leqq \frac{2}{[\log R_2/R_1]^2}\int_{R_1}^{R_2}\frac{p(R)dR}{R}. \tag{3.5}$$

However, eventual mean p-valence implies

$$\begin{aligned}\int_{R_1}^{R_2}\frac{p(R)dR}{R} &= p\log\frac{R_2}{R_1} + \int_{R_1}^{R_2}\frac{h(R)\,dR}{R}\\ &= p\log\frac{R_2}{R_1} + \frac{H^*(R_2)}{2R_2^2} - \frac{H^*(R_1)}{2R_1^2} + \int_{R_1}^{R_2}\frac{H^*(R)\,dR}{R^3}\\ &\leqq p\left[\log\frac{R_2}{R_1} + \frac{1}{2}\right],\end{aligned} \tag{3.6}$$

since $-pR^2 \leqq H^*(R) \leqq 0$ for all $R \geqq R_0$. Therefore, using (3.5) and (3.6), we obtain our desired conclusion.

DEFINITION. ([6, p. 34].) Let $f(z)$ be analytic in $|z| < 1$. Suppose that for $\zeta = \exp(i\theta)$ there is a path $\gamma(\theta)$ lying, except for its endpoint ζ, in $|z| < 1$, and also a positive δ such that

$$\liminf\,(1 - |z|)^{\delta}|f(z)| > 0$$

as $z \to \zeta$ along $\gamma(\theta)$. Then the order $\alpha(\zeta)$ of $f(z)$ at ζ is defined to be the upper bound of all such δ's. If no path $\gamma(\theta)$ and positive δ exist, we put $\alpha(\zeta) = 0$.

THEOREM 5. ([6, Th. 2.7].) *If $f(z)$ is eampv in D with respect to R_0, then the set E of distinct points ζ on the boundary of D for which $\alpha(\zeta) > 0$ is countable and satisfies $\Sigma_E \alpha(\zeta) \leqq 2p$.*

PROOF. One possible proof follows [6] where Theorem 4 replaces [6, Th. 2.6]; we no longer have to worry about the zeros of $f(z)$. An interesting alternate proof uses a ploy one wishes would work more often. It is easy to show that for any $\varepsilon > 0$ and any *eampv* function $f(z)$, one can find a complex number b such that $g(z) = f(z) + b$ has only a finite number of zeros and $f(z)$ is *eam*$(p+\varepsilon)v$ with respect to $R = R(p, R_0)$. (See Theorem 14.) One then applies [6, Th. 2.7] directly to $g(z)$ (which has the same set E as does $f(z)$) to obtain $\Sigma_E \alpha(\theta) \leqq 2(p+\varepsilon)$. But ε is arbitrary and therefore $\Sigma_E\ \alpha(\theta) \leqq 2p$. Unfortunately, this line of attack appears to fail for Theorems 3, 6, and 7 as well as for other interesting situations.

The next theorem is the only real disappointment to our general theme; we conjecture that it is true without the restrictions on the number of zeros of $f(z)$. Nevertheless it is still a nontrivial generalization of Hayman's theorem since (i) it applies to functions *which are not of locally finite area* and (ii) the number of zeros at the origin can be arbitrarily large.

THEOREM 6. ([6, Th. 2.8].) *Let $f(z)$ be eampv in D with respect to R_0 and have only a finite number of zeros in D. If*

$$\limsup_{r \to 1} (1-r)^{2p} M(r,f) = \alpha > 0,$$

then there exists a θ_0 $(0 \leqq \theta_0 \leqq 2\pi)$ such that

$$\alpha_0 = \liminf_{r \to 1} (1-r)^{2p} \left| f(r\exp(i\theta_0)) \right| \geqq \frac{\alpha}{A(f, R_0)} > 0. \tag{3.7}$$

PROOF. Identical to [6, Lem. 2.4] and Lemma 1.

Eke [4] has recently shown for *ampv* functions that the existence of $\limsup\ (1-r)^{2p}\ M(r,f) = \alpha > 0$ implies the existence of a unique θ such that $\lim_{r \to 1}\ (1-r)^{2p}\ \left|f(r\exp(i\theta))\right| = \alpha$. The reader is referred to his beautiful paper with the forewarning that his results require considerable analytic machinery.

From Theorem 5 it is clear that there is at most one point θ_0 for which (3.7) can hold. Therefore whenever such a θ_0 exists we have the following theorem.

THEOREM 7. ([6, Th. 2.9].) *Let $f(z)$ be eampv in D with respect to R_0. If there is a $\theta_0 \in [0, 2\pi)$ such that*

$$\liminf_{r \to 1} (1 - r)^{2p} |f(r\exp(i\theta_0))| > 0$$

then for any ε, $0 < \varepsilon < 2p$, we can find a positive constant C and an r_0 such that

$$|f(r\exp(i\theta))| < (1 - r)^{-\varepsilon} |\theta - \theta_0|^{\varepsilon - 2p} \tag{3.8}$$

for all r, $r_0 < r < 1$, $C(1 - r) \leqq |\theta - \theta_0| \leqq \pi$.

Furthermore, we have uniformly as $r \to 1$ while $\varepsilon \leqq |\theta - \theta_0| \leqq \pi$

$$\log |f(r\exp(i\theta))| < O\left[\log \frac{1}{1 - r}\right]^{\frac{1}{2}}. \tag{3.9}$$

PROOF. Let $\alpha_0 = \liminf_{r \to 1} (1 - r)^{-2p} |f(r\exp(i\theta_0))| > 0$. Then there is a $\delta > 0$ such that

$$|f(r\exp(i\theta_0))| \geqq \tfrac{1}{2}\alpha_0 (1 - r)^{-2p} > R_0 \tag{3.10}$$

for all $r, 1 - \delta < r < 1$. Let $z_1 = (1 - \delta)\exp(i\theta_0)$, $z_2 = (1 - \delta)\exp(i\theta)$, and assume that $4\delta \leqq |\theta - \theta_0| \leqq \pi$. The discs $|z - z_1| < \delta$ and $|z - z_2| < \delta$ are, of course, disjoint subsets of D.

We wish to apply Theorem 4 to the discs $|z - z_1| < \delta$, $|z - z_2| < \delta$. Therefore, since $|f(z_1)| > R_0$, the value of R_1 in Theorem 4 is

$$R_1^* = \max_{j=1,2} \max_{|\zeta| \leqq \tanh \frac{1}{2}\pi} |f(z_j + \delta\zeta)|.$$

A quick glance at the proof of Theorem 4 will convince the reader that the conclusion still holds if we also demand that $R_1 = \max[R_1^*, M(1 - \delta, f)]$. Clearly by Theorem 3, $R_1 \leqq A(p, R, f)\,(\tanh \frac{1}{2}\pi)^{-2p}\,\delta^{-2p}$.

Suppose that

$$|f(r_2\exp(i\theta)| = R_2 > eR_1 \tag{3.11}$$

where $1 - \delta < r_2 < 1$. Let $z_2' = r_2\exp(i\theta)$ and choose $z_1' = r_1\exp(i\theta_0)$ such that r_1 is the smallest number for which $|f(z_1')| = |f(r_1\exp(i\theta_0)| = R_2$. Such a number exists by (3.10) and (3.11). Of course $r_1 > 1 - \delta$ from the definition of R_1.

We now apply Theorem 4 with

$$\delta_1 = \frac{\delta - [r_1 - (1 - \delta)]}{\delta} = \frac{1 - r_1}{\delta}, \quad \delta_2 = \frac{1 - r_2}{\delta},$$

and obtain

$$(3.12)\qquad \left[\log\left(\frac{2e^{\pi}\delta}{1-r_1}\right)\right]^{-1} \leqq \frac{2p}{\log(R_2/R_1)-1} - \left[\log\frac{2e^{\pi}\delta}{1-r_1}\right]^{-1}.$$

By (3.10), $R_2 \geqq \frac{1}{2}\alpha_0(1-r_1)^{-2p}$ and by Theorem 3, as previously remarked, $R_1 \leqq A(p,f,R_0)\delta^{-2p}$. Therefore $R_2/R_1 \geqq C_1[\delta/(1-r_1)]^{2p}$, where C_1 is a constant depending on f, p, and R_0. The remainder of the proof is now identical to [6] and is omitted.

As Hayman states, condition (3.9) is a good deal stronger than the condition that $\alpha(\zeta) = 0$ for all $\exp(i\theta) \neq \exp(i\theta_0)$.

The next series of theorems concentrate on the rate of growth of *eampv* functions which have one of three characteristics: some form of symmetry, gaps in the power series development, or some control of growth of the coefficients on an arithmetic sequence of coefficients. The key is the following technical result.

THEOREM 8. ([6, Th. 3.7].) *Suppose that $f(z)$ is eampv in D with respect to R_0. If for all r sufficiently close to one, there exist $k \geqq 2$ points $z'_1, z'_2, \cdots, z'_k$ on $|z| = r$ such that*

(i) $|z'_i - z'_j| \geqq \delta$ *(for $1 \leqq i < j \leqq k$, where δ is independent of r) and*

(ii) $|f(z'_i)| \geqq R$ *(for $1 \leqq i \leqq k$),*

then

$$(3.13)\qquad R < A(p,f,R_0)\delta^{2p(1/k-1)}(1-r)^{-2p/k}.$$

PROOF. We may suppose that

$$(3.14)\qquad \delta > 4^{p+2}(1-r)$$

for if this is not the case then, by Theorem 3,

$$\begin{aligned} R \leqq M(r,f) &< A(p,f,R_0)\,(1-r)^{-2p} \\ &< A(p,f,R_0)\,(1-r)^{-2p/k}(4^{p+2}/\delta)^{2p-2p/k} \\ &< A(p,f,R_0)(1-r)^{-2p/k}\delta^{2(p/k)-2p}, \end{aligned}$$

which concludes the proof of Theorem 8 in this case. Therefore, letting $r_1 = 1 - \frac{1}{8}\delta = 1 - \delta_0$ and noting $\frac{1}{8}\delta = \delta_0 > 4^{p+2}\frac{1}{8}(1-r) \geqq 2(1-r)$, we conclude that $r_1 < r$. Let $z'_j = r\exp(i\theta_j)(1 \leqq j \leqq k)$ and define $z_j = r_1\exp(i\theta_j)$ $(1 \leqq j \leqq k)$. By construction, the disc $|z - z_j| < \delta_0$ contains the point z'_j and has diameter $2\delta_0 = \frac{1}{4}\delta$, which, because of hypothesis (i), implies that the discs $|z - z_j| < \delta_0$ $(1 \leqq j \leqq k)$ are nonoverlapping. Now let $R_2 = R$ and $R_1 = \max\{R_0, \max\{|f(z)| : |z| = r_1 + \delta_0 \tanh\frac{1}{4}\pi\}\}$. We let $r_2 = r_1 + \delta_0 \tanh\frac{1}{4}\pi$ and note that

$r_2 = 1 - \frac{1}{8}\delta\,(1 - \tanh \frac{1}{4}\pi) = 1 - .0434\,\delta$. Thus r_2 is independent of r and strictly less than one. Consequently, for r sufficiently close to one, $r_2 < r < 1$. Finally, we may suppose that $R_2 \geqq eR_1$ since otherwise the conclusion is obvious (R_1 is independent of r).

Thus if we set

$$\delta_j = \frac{(\delta_0 - |z_j' - z_j|)}{\delta_0} = \frac{(1-r)}{\delta_0} \text{ for } 1 \leqq j \leqq k,$$

we may apply Theorem 4 (with an obvious modification) and obtain

$$k\left[\log\left(\frac{2e^{\pi}\delta_0}{1-r}\right)\right]^{-1} \leqq 2p\,[\log(R_2/eR_1)]^{-1}, \text{ hence}$$

$$\frac{R_2}{eR_1} < \left(\frac{2e^{\pi}\delta_0}{1-r}\right)^{2p/k} < A(p)\left(\frac{\delta}{1-r}\right)^{2p/k}, \tag{3.15}$$

since $\delta_0 = \frac{1}{8}\delta$. However $r_1 = 1 - \delta_0$ and an application of Theorem 3 yields

$$R_1 = M(r_1, f) < A(p, f, R_0)\,\delta_0^{-2p} = A(p, f, R_0)\,\delta^{-2p}. \tag{3.16}$$

Theorem 8 follows from (3.15) and (3.16) upon writing R instead of R_2.

If an *eampv* function eventually shows a form of k-fold symmetry then, as in the case for *ampv* functions, the growth of $M(r,f)$ is severely restricted.

THEOREM 9. ([6, Th. 3.8].) *Suppose that*

$$f(z) = \sum_{j=0}^{N-1} a_j z^j + \sum_{j=0}^{\infty} a_{N+jk} z^{N+jk}$$

is eampv in D with respect to R_0. *Then*

$$M(r,f) < A(p, k, N, f, R_0)\,(1-r)^{-2p/k} \text{ for } 0 < r < 1.$$

PROOF. The proof follows that of [6, Th. 3.8] where we replace [6, Th. 3.7] by Theorem 8 and note that

$$\left| \sum_{j=0}^{N-1} a_j z^j \right| \leqq A(N,f)$$

for all $|z| < 1$, where $A(N,f)$ is a constant depending only on N and f.

If an *eampv* function only has coefficient gaps we can still make some strong assertions.

THEOREM 10. ([6, Th. 3.9].) *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is eampv in D with respect to* R_0. *If* $a_n = 0$ *whenever* $n = bm + c$, *where b and c are fixed positive integers and m goes from* 1 *to* ∞, *then*

$$M(r,f) < A(p,b,c,f,R_0)(1-r)^{-p} \text{ for } 0 < r < 1.$$

PROOF. The proof follows that of [6, Th. 3.9] with the stronger Theorem 8 replacing [6, Th. 3.7] of Hayman.

We now generalize a theorem due to Goluzin [5, p. 190] for which, when p is greater than or equal to one, Theorem 10 is but a special case with $\alpha = p - 1$.

THEOREM 11. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be eampv in D with respect to R_0. Let b and c be positive integers with $1 \leqq c \leqq b$. Suppose that there is a positive integer N such that for all integers $n = bm + c$, with m an integer $\geqq N$, we have $|a_n| \leqq Cn^{\alpha}$ where $\alpha \geqq \max\{p-1, 0\}$. Then*

$$M(r,f) < A(p,b,f,N,R_0)(1-r)^{-(\alpha+1)}.$$

We shall see that the restriction $\alpha \geqq \max\{p-1,0\}$ is necessary and cannot be removed for any $p \geqq 1$. A discussion of this is presented at the end of the paper.

PROOF. We begin by writing $g(z) = \sum_{n=0}^{N-1} a_n z^n$, $h(z) = \sum_{n=N}^{\infty} a_n z^n$ and let $h_v(z) = \sum_{m=0}^{\infty} a_{bm+v} z^{bm+v}$ $(0 \leqq v \leqq b-1)$. Then $h(z) = \sum_{v=0}^{b-1} h_v(z)$. Moreover,

$$\begin{aligned} |h_c(z)| &\leqq \sum_{m=0}^{\infty} |a_{bm+c}| r^{bm+c} \leqq C r^c \sum_{m=0}^{\infty} (bm+c)^{\alpha} r^{bm} \\ &\leqq b^{\alpha} C r \sum_{m=0}^{\infty} (m+1)^{\alpha} (r^b)^m \\ &\leqq C b^{\alpha} r \left(1 + 2^{\alpha} \sum_{m=0}^{\infty} m^{\alpha} (r^b)^m\right) \end{aligned}$$

from which, as is well known [5, p. 191], it follows that

$$|h_c(z)| \leqq A(\alpha,b,c,C)(1-r)^{-(1+\alpha)}. \tag{3.17}$$

Now select and fix an arbitrary z of modulus r for $0 < r < 1$. Consider the maximum of $|f(z\exp(2\pi ij/b))|$ such that $0 \leqq j \leqq b-1$. Without loss of generality, we may assume that the maximum term occurs for $j = 0$. Then applying Theorem 8 to $|f(z)|$ and $|f(z\exp(2\pi ij/b))|$, we have for each $j = 1, \cdots, b-1$,

$$|f(z\exp(2\pi ij/b))| < A(p,f,R_0)\,|1-\exp(2\pi ij/b)|^{-p}|(1-r)^{-p}.$$

In particular,

$$|f(z\exp(2\pi ij/b))| < A(p,f,R_0,b)(1-r)^{-p} \qquad j = 1, \cdots, b-1.$$

But,

$$f(z\exp(2\pi ij/b)) = g(z\exp(2\pi ij/b)) + \sum_{v=0}^{b-1} \exp(2\pi ijv/b)h_v(z).$$

Therefore, we have for all $j = 1, \cdots, b-1$,

$$\left|g(z\exp(2\pi ij/b)) + \sum_{v=0}^{b-1} \exp(2\pi ij/b)\, h_v(z)\right| < A(p, R_0, b)\ (1-r)^{-p}$$

from which we deduce for $j = 1, \cdots, b-1$

$$\left|\sum_{v=0}^{b-1} \exp(2\pi ijv/b)\, h_v(z)\right| < A(p, f, b, R_0, N)\ (1-r)^{-p}.$$

Let $w \neq c$ be an integer satisfying $0 \leqq w \leqq b-1$. If we consider the $b-1$ equations

$$\left|\sum_{v=0}^{b-1} A_j \exp(2\pi ijv/b) h_v(z)\right| < |A_j| A(p, f, b, R_0, N)\ (1-r)^{-p},$$

where $j = 1, \cdots, b-1$, and add them, we obtain

$$\left|\sum_{v=0}^{b-1}\left(\sum_{k=1}^{b-1} A_k \exp(2\pi ikv/b)\right) h_v(z)\right| < \left(\sum_{j=1}^{b-1} |A_j|\right) A(p, f, b, R_0, N)(1-r)^{-p}. \tag{3.18}$$

However, we can always solve the system of equations

$$\sum_{k=1}^{b-1} A_k \exp(2\pi ikv/b) = \delta_{vw},\ 0 \leqq v \leqq b-1,\ v \neq c, \tag{3.19}$$

since the minors of the b by b Vandermonde determinant created from the numbers $\exp(2\pi ik/b)$ $(k = 1, \cdots, b)$ are all nonzero (see [5, pp. 190–2]). Consequently from (3.18) and (3.19) we can always write

$$\begin{aligned} &\left|h_w(z) + h_c(z)\left(\sum_{k=1}^{b-1} A_k \exp(2\pi ikc/b)\right)\right| \\ &< \left(\sum_{j=1}^{b-1} |A_j|\right) \cdot A(p, f, N, b, R_0)\ (1-r)^{-p}. \end{aligned} \tag{3.20}$$

In particular, from (3.17), (3.20) and the fact that $\alpha \geqq \max\{0, p-1\}$, we obtain for any $w \neq c$, $0 \leqq w \leqq b-1$ that

$$|h_w(z)| < A(p, f, N, b, R_0)\ (1-r)^{-(\alpha+1)}\ . \tag{3.21}$$

Therefore, using (3.17) and (3.21), we have

$$\begin{aligned} |f(z)| &\leqq |g(z)| + \left|\sum_{v=0}^{b-1} h_v(z)\right| \\ &\leqq A(f, N) + A(p, f, N, b, R_0)\ (1-r)^{-(\alpha+1)} \\ &< A(p, f, N, b, R_0)\ (1-r)^{-(\alpha+1)}. \end{aligned}$$

Since z was arbitrary on $|z| = r$ we have $M(r,f) < A(p,f,N,b,R_0)(1-r)^{-(\alpha+1)}$ and the proof is complete.

4. Coefficient estimates of *eampv* functions

We now present the Hardy-Spencer-Stein machinery for *eampv* functions. As in [6] we let

$$S_\lambda(r,f) = r\frac{d}{dr}I_\lambda(r,f) = r\frac{d}{dr}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(r\exp(i\theta)|^\lambda d\theta\right\}$$

and

$$p(r,R) = p(r,R,f) = \frac{1}{2\pi}\int_0^{2\pi} n(r,R\exp(i\psi))d\psi$$

where $n(r,R\exp(i\psi))$ denotes the number of roots of $f(z) = R\exp(i\psi)$ in $|z| < r$.

THEOREM 12. ([6, Th. 3.2].) *Let $f(z)$ be eampv in D with respect to R_0. Suppose that there is an $R_1 > 0$ such that for all R, $0 < R \leqq R_1$, we have $W(R) \leqq qR^2$. Let $\Lambda = \max(\lambda, \frac{1}{2}\lambda^2)$ for $\lambda > 0$ and $\Lambda^* = \frac{1}{2}R^{\lambda-2}$ if $\lambda > 2$; $\Lambda^* = \frac{1}{2}R_1^{\lambda-2}$ if $0 < \lambda \leqq 2$. Then*

$$S_\lambda(r,f) \leqq p\Lambda M^\lambda(r,f) + q\Lambda R_1^\lambda + \Lambda^* A(r,R_0) \tag{4.1}$$

and

$$I_\lambda(r,f) \leqq M^\lambda(r_0,f) + \int_{r_0}^{r}\left\{\frac{p\Lambda M^\lambda(r,f) + q\Lambda R_1^\lambda + \Lambda^* A(r,R_0)}{r}\right\}dr \tag{4.2}$$

for $0 < r_0 < r < 1$ where $A(r,R_0)$ denotes the surface area of $f(|z| \leqq r)$ over $|w| \leqq R_0$.

PROOF. From the Hardy-Spencer-Stein identity for an arbitrary analytic function in D we have [6, p. 42]

$$\begin{aligned} S_\lambda(r,f) &= \lambda^2\int_0^\infty p(r,R)R^{\lambda-1}dR = \lambda^2\int_0^{M(r,f)} p(r,R)R^{\lambda-1}dR \\ &= \lambda^2\left\{\int_0^{R_1} + \int_{R_1}^{R_0} + \int_{R_0}^{M(r,f)}\right\}p(r,R)R^{\lambda-1}dR \end{aligned} \tag{4.3}$$

for all r, $0 < r < 1$. We now estimate each of the three integrals separately.

Since $f(z)$ is initially *amqv* for $0 \leqq R \leqq R_1$, then exactly as in the proof of [6, Theorem 3.2], we can conclude that

$$\int_0^{R_1} p(r,R)R^{\lambda-1}dR \leqq q\Lambda R_1^\lambda \tag{4.4}$$

for all r, $0 < r < 1$.

Next we note that if $R_0 \geqq R_1$

$$\begin{aligned} \int_{R_1}^{R_0} p(r,R)R^{\lambda-1}dR &= \tfrac{1}{2}\int_{R_1}^{R_0} R^{\lambda-2}2p(r,R)\,RdR \\ &\leqq \Lambda^* \int_{R_1}^{R_0} p(r,R)\,d(R^2) \qquad (4.5) \\ &\leqq \Lambda^* \int_{0}^{R_0} p(r,R)\,d(R^2). \end{aligned}$$

This last integral is the area of the image of $f(|z| \leqq r)$ (considered as a surface) lying over $|w| \leqq R_0$.

If r is such that $M(r,f) \leqq R_0$, then $\int_{R_0}^{M(r,f)} p(r,R)\,R^{\lambda-1}dR \leqq 0$ and $S_\lambda(r,f) \leqq q\Lambda R_1^\lambda + \Lambda^* A(r,R_0)$ which is trivially less than or equal to $p\Lambda M^\lambda(r,f) + q\Lambda R_1^\lambda + \Lambda^* A(r,R_0)$ as claimed in (4.1).

We therefore proceed to evaluate the remaining integral under the assumption that $M(r,f) > R_0$. Setting

$$\int_{R_0}^{R} p(r,\rho)\,d(\rho^2) = W^*(r,R)$$

and $M = M(r,f)$ we obtain

$$\begin{aligned} \int_{R_0}^{M} p(r,R)R^{\lambda-1}\,dR &= \tfrac{1}{2}\int_{R_0}^{M} R^{\lambda-2}\frac{d}{dR}\,W^*(r,R) \\ &= \tfrac{1}{2}R^{\lambda-2}W^*(r,R)\Big|_{R_0}^{M} + \frac{2-\lambda}{2}\int_{R_0}^{M} R^{\lambda-3}W^*(r,R)\,dR. \end{aligned}$$

The fact that $f(z)$ is *eampv* implies for all $R \geqq R_0$

$$pR^2 \geqq W^*(1,R) \geqq W^*(r,R) \geqq W^*(r,R_0) = 0.$$

Hence, if $\lambda > 2$

$$\begin{aligned} \int_{R_0}^{M} p(r,R)\,R^{\lambda-1}dR &\leqq \tfrac{1}{2}M^{\lambda-2}W^*(r,M) \\ &\leqq \tfrac{1}{2}M^{\lambda-2}\,pM^2 = \tfrac{1}{2}p\,M^\lambda(r,f), \end{aligned}$$

while if $0 < \lambda \leqq 2$, then

$$\begin{aligned} \int_{R_0}^{M} p(r,R)\,R^{\lambda-1}dR &\leqq \tfrac{1}{2}M^{\lambda-2}p\,M^2 + \frac{2-\lambda}{2\lambda}\int_{R_0}^{M} R^{\lambda-3}pR^2dR \\ &\leqq \tfrac{1}{2}p\,M^\lambda + \frac{p(2-\lambda)}{2\lambda}M^\lambda = \frac{p}{\lambda}M^\lambda(r,f). \end{aligned}$$

Thus in both cases we obtain $\int_{R_0}^{M} p(r,R) R^{\lambda-1} dR \leqq p\Lambda M^{\lambda}(r,f)$. Combining these three estimates we obtain (4.1). Clearly, for any r, $0 < r_0 < r < 1$ we have

$$\begin{aligned} I_{\lambda}(r,f) &= I_{\lambda}(r_0,f) + \int_{r_1}^{r} \frac{S_{\lambda}(t,f)}{t} dt \\ &\leqq M^{\lambda}(r_0,f) + \int_{r_1}^{r} \left\{ \frac{p\Lambda M^{\lambda}(r,f) + q\Lambda R_1^{\lambda} + \Lambda^{*} A(r,R_0)}{r} \right\} dr \end{aligned}$$

as claimed in (4.2). This completes the proof of the theorem.

We now establish a technical lemma which is necessary to relate $M(r,f)$ to coefficient growth.

LEMMA 13. ([6, Lem. 3.1].) *Suppose* $0 < \lambda \leqq 2$ *and* $f(z)$ *satisfies the conditions of Theorem* 12. *If* $\frac{5}{8} < r < 1$, *then there is a* ρ *such that* $2r - 1 \leqq \rho \leqq r$ *and*

$$\tag{4.6} \begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho \exp(i\theta))|^2 \, |f(\rho \exp(i\theta))|^{\lambda-2} d\theta \\ &\leqq \frac{4}{1-r} \left\{ \frac{p}{\lambda} M^{\lambda}(r,f) + \frac{q}{\lambda} R_1^{\lambda} + \frac{R_1^{\lambda-2}}{2\lambda^2} \cdot A(r,R_0) \right\}. \end{aligned}$$

PROOF. From the Hardy-Spencer-Stein equality we have

$$\frac{1}{2\pi} \int_0^{r} \rho \, d\rho \int_0^{2\pi} |f'(\rho \exp(i\theta))|^2 |f(\rho \exp(i\theta))|^{\lambda-2} d\theta = \lambda^{-2} S_{\lambda}(r,f)$$

and thus by Theorem 12

$$\begin{aligned} &\frac{1}{2\pi} \int_{2r-1}^{r} \rho d\rho \int_0^{2\pi} |f'(\rho \exp(i\theta))|^2 |f(\rho \exp i\theta))|^{\lambda-2} d\theta \\ &\leqq \frac{p}{\lambda} M^{\lambda}(r,f) + \frac{q}{\lambda} R_1^{\lambda} + \frac{R_1^{\lambda-2}}{2\lambda^2} \cdot A(r,R_0) \end{aligned}$$

for $0 < r < 1$. By the mean value theorem for integrals there exists a ρ such that $2r - 1 \leqq \rho \leqq r$ and

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho \exp(i\theta)|^2 \, | f(\rho \exp(i\theta))|^{\lambda-2} d\theta \\ &\leqq \frac{1}{(1-r)\rho} \left[\frac{p}{\lambda} M^{\lambda}(r,f) + \frac{q}{\lambda} R_1^{\lambda} + \frac{R_1^{\lambda-2}}{2\lambda^2} \cdot A(r,R_0) \right] \\ &\leqq \frac{4}{(1-r)} \left[\frac{p}{\lambda} M^{\lambda}(r,f) + \frac{q}{\lambda} R_1^{\lambda} + \frac{R_1^{\lambda-2}}{2\lambda^2} \cdot A(r,R_0) \right]. \end{aligned}$$

The last inequality follows since $\frac{1}{4} \leqq 2r - 1 < \rho$.

THEOREM 14. ([6, Th. 3.3].) *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be eampv in D with respect to* R_0. *If*

$$M(r,f) \leqq C(1-r)^{-\alpha} \tag{4.7}$$

for $0 < r < 1$ *where* $C > 0$ *and* $\alpha > \frac{1}{2}$ *and if*

$$A(r, R_0) = C'(1-r)^{-\gamma} \tag{4.8}$$

where $C' > 0$ *and* $\gamma \geqq 0$, *then*

$$|a_n| \leqq A(C, C', R_0, f, p, \alpha, \gamma) n^{\omega - 1}$$

where

$$\omega = \begin{cases} \gamma & , \text{ if } \alpha + \frac{1}{2} \leqq \gamma \\ \frac{1}{2}\alpha + \frac{1}{2}\gamma + \frac{1}{4}, & \text{ if } \alpha - \frac{1}{2} \leqq \gamma \leqq \alpha + \frac{1}{2} \\ \alpha & , \text{ if } 0 \leqq \gamma \leqq \alpha - \frac{1}{2} \end{cases} .$$

PROOF. We begin by showing there is a complex number c such that $f(z) + c$ is initially *amqv* for some finite q. Let $A(c, R)$ denote the surface area of $f(D)$ lying over the open disc $|w - c| < R$. If there is a complex number c such that

$$\limsup_{R \to 0} \frac{A(c, R)}{R^2} < \infty \tag{4.9}$$

then the function $f(z) + c$ will be initially *amqv* for some $q < \infty$ (that is, there will be an $R_1 > 0$ such that $W(R, f(z) + c) \leqq qR^2$ for all $0 < R \leqq R_1$). If (4.9) were false for all $|c| \geqq R_0$, then for every point c, $|c| \geqq R_0$, we can find an open disc centered at c of radius R, $R < R_0$, for which $A(c, R) \geqq 3\pi pR^2$, that is $A(c, R)/30p \geqq \pi R^2$. This creates an open covering for the compact annulus $2R_0 \leqq |w| \leqq 3R_0$. There is therefore a finite subcover of this annulus and this subcover lies in the annulus $R_0 \leqq |w| \leqq 4R_0$. By the Vitali covering theorem, we may choose a finite subcollection of this finite subcover consisting of mutually disjoint discs $|w - c_k| < R_k$ whose collective area, $\sum \pi R_k^2$, is at least $5\pi R_0^2/9$ (where $5\pi R_0^2$ is the area of the annulus $2R_0 \leqq |w| \leqq 3R_0$). Since the surface area of $f(D)$ over the disc $|w - c_k| < R_k$ satisfies $A(c_k, R)/30p \geqq \pi R_k^2$, we see that the area of the surface over this finite disjoint collection of discs is at least $150\pi pR_0^2/9$. On the other hand, since the discs are disjoint and contained in $R_0 \leqq |w| \leqq 4R_0$, the assumption that $f(z)$ is *eampv* with respect to R_0 implies that the surface area over these discs can be at most $16\pi p R_0^2 = 144\pi p R_0^2/9$. This contradiction shows that there is a $|c| \geqq 2R_0$ such that $f(z) + c$ is initially *amqv*.

It is easy to check that if $f(z)$ is *eampv* with respect to R_0, then for any $c \in \mathbb{C}$ and any $\varepsilon > 0$, there is an $R_1 = R_1(R_0, \varepsilon, c)$ for which the function $f(z) + c$ is *eam*$(p+\varepsilon)v$ with respect to R_1. If $f(z)$ is *eampv* and $A(r, R_0) \leqq C'(1-r)^{-\gamma}$, it is easy to verify that $A(r, R_1, f(z)+c) \leqq C'(R_0, p, c)\,(1-r)^{-\gamma}$. Finally, if $c \in \mathbb{C}$ and $M(r,f) \leqq C(1-r)^{-\alpha}$ then $M(r, f(z)+c) \leqq (C + |c|)\,(1-r)^{-\alpha}$.

Since the coefficient estimates for $f(z)$ and $g(z) = f(z) + c$ are the same, we will concentrate on $g(z)$ where (i) $g(z)$ is initially *amqv* for $0 < R \leqq R_1$, $q < \infty$, (ii) $g(z)$ is *eam*$(p+1)v$ for $R_0^* = R_0(R_0, f)$, (iii) $M(r,g) \leqq C(C,f)\,(1-r)^{-\alpha}$, and (iv) $A(r, R_0^*, g) \leqq C'(C', R_0, f, p)(1-r)^{-\gamma}$.

We now suppose that $r \geqq \frac{5}{8}$, and $\alpha > \frac{1}{2}$. Let $\lambda = (2\alpha - 1)/2\alpha$ so that $\alpha(2-\lambda) > 1$ and choose ρ so that (4.6) holds. Then

$$(4.10)\qquad I_1(\rho, g') = \frac{1}{2\pi}\int_0^{2\pi} |g'(\rho\exp(i\theta))|\,d\theta$$
$$\leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |g'(\rho\exp(i\theta))|^2\,|g(\rho\exp(i\theta))|^{\lambda-2}\,d\theta\right)^{\frac{1}{2}}$$
$$\cdot\left(\frac{1}{2\pi}\int_0^{2\pi} |g(\rho\exp(i\theta))|^{2-\lambda}\,d\theta\right)^{\frac{1}{2}}$$

by Schwarz's inequality. Letting $r_0 = \frac{5}{8}$ in (4.2) and noting that $r \geqq \frac{5}{8}$, $r \geqq \rho$, we deduce

$$I_{2-\lambda}(\rho, g) \leqq I_{2-\lambda}(r, g)$$
$$\leqq (C+|c|)(1-r_0)^{-\alpha(2-\lambda)} + \frac{1}{r_0}\int_0^r \{(p+1)\Lambda M^{2-\lambda}(r,g) + q\Lambda R_1^{2-\lambda} + \Lambda^* A(r,g)\}\,dr$$
$$\leqq A(C, C', p, \alpha, R_0, f, \gamma)\int_0^r \{(1-r)^{-\alpha(2-\lambda)} + (1-r)^{-\gamma}\}\,dr$$
$$\leqq A(C, C', p, \alpha, \gamma, R_0, f)\,(1-r)^{-\mu+1}$$

where $\mu = \max(\gamma, \alpha + \frac{1}{2})$. By Lemma 13

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |g'(\rho\exp(i\theta))|^2\,|g(\rho\exp(i\theta))|^{\lambda-2}\,d\theta\right\}^{\frac{1}{2}}$$
$$\leqq (1-r)^{-\frac{1}{2}} A(C, C', p, \alpha, \gamma, R_0, f)(1-r)^{-\nu/2}$$

where $\nu = \max(\gamma, \alpha - \frac{1}{2})$. (Note that we rely on the fact that R_1, q, and c depend only on f.) Now if we write $r_1 = 2r - 1$ so that $r_1 \leqq \rho < r$, then we may deduce from (4.6), (4.10), and the above that

$$I_1(r_1, g') \leqq I_1(\rho, g') \leqq A(C, C', p, \alpha, \gamma, R_0, f)\,(1-r)^{-\frac{1}{2}[\mu+\nu]}.$$

The conclusion follows with the observation that

$$\mu + \nu = \begin{cases} 2\gamma & \text{if } \alpha + \frac{1}{2} \leqq \gamma \\ \alpha + \frac{1}{2} + \gamma & \text{if } \alpha - \frac{1}{2} \leqq \gamma \leqq \alpha + \frac{1}{2} \\ 2\alpha & \text{if } 0 \leqq \gamma \leqq \alpha - \frac{1}{2} \end{cases}$$

and the fact that

$$|a_n| \leqq \frac{e}{n} I_1\left(\frac{n}{n+1}, f'\right).$$

A beautiful result of univalent function theory concerns the rate of growth of $M(r,f)$ and the rate of growth of the coefficients of $f(z)$. It is well known that for functions $f(z) = \Sigma a_n z^n$ which are *ampv* (hence for all p-valent functions in particular), if $\alpha > \frac{1}{2}$ then $M(r,f) = O(1-r)^{-\alpha}$ implies $|a_n| = O(n^{\alpha-1})$. The breakdown at $\alpha = \frac{1}{2}$ is not because of a limitation in the method of proof. (Indeed Littlewood [7] has given an example of a bounded univalent function for which $|a_n| > n^{\sigma-1}$ for some positive σ for infinitely many n.) One may ask for which classes is it true that the estimate $M(r,f) = O(1-r)^{-\alpha}$ implies $|a_n| = O(n^{\alpha-1})$ for various restrictions on α. Clunie and Pommerenke [3] have shown that for close-to-convex functions one may let $\alpha \geqq 0$. This was extended in my dissertation [1] to a large class of locally univalent functions which contain as subclasses the close-to-convex functions and the functions of bounded boundary rotation (V_k). Now I will show that under various additional restrictions on either $f(z)$ or on α a similar phenomenon can occur for *eampv* functions. Let us briefly summarize a particular function class.

Pommerenke [8] introduced and investigated an extremely important and natural generalization of the normalized univalent functions, namely the classes $\mathfrak{U}_\beta$, $\beta \geqq 1$. A function is in $\mathfrak{U}_\beta$ if and only if

$$\sup_{z \in D} \left| -\bar{z} + \tfrac{1}{2}(1-|z|^2) f''(z)/f'(z) \right| \leqq \beta.$$

We let $X = \bigcup \{\mathfrak{U}_\beta : \beta \geqq 1\}$ be the set of all locally univalent analytic functions of finite order. All functions in $\mathfrak{S}_p$ (the locally univalent functions which are globally at most p-valent), the functions in V_k (bounded boundary rotation), the functions of bounded argument ($\sup|\arg f'(z)| < \infty$), all of these and many other classical geometric function theory classes can be dealt with in a systematic manner by the general theory of locally univalent functions of finite order developed by Pommerenke. Clearly functions in $\mathfrak{U}_\beta$ satisfy

$$\operatorname{Re}(1 + zf''(z)/f'(z)) \geqq (1 - 2\beta r + r^2)/(1 - r^2).$$

It is well known that if p is a positive integer and

$$\operatorname{Re}(1 + zf''(z)/f'(z)) > -\tfrac{1}{2}p$$

for $|z| < r$, then $f(z)$ is at most p-valent in $|z| < r$. Consequently, if $f(z) \in \mathfrak{U}_\beta$, then for r close to one we see that $f(z)$ is at most $4\beta/(1-r)$-valent in $|z| < r$. Thus the area of $f(|z| < r)$ over any disc of radius R_0 is no more than $4\beta\pi R_0^2/(1-r)$. In particular, if $f(z) \in \mathfrak{U}_\beta$, $A(r,f,R_0) \leqq 4\beta\pi R_0^2(1-r)^{-1}$.

The theorems that remain to be stated result from immediate application of Theorem 14, from $M(r,f)$ restrictions which were deduced in Section 3, and from application of the fact that $A(r,R_0) = O(1-r)^{-1}$ for $f \in X$.

THEOREM 15. ([6, Th. 3.5].) *Let $f(z)$ be eampv in D with respect to R_0. If $A(r,R_0) = O(1-r)^{-\gamma}$, $0 \leqq \gamma \leqq 2p - \frac{1}{2}$, $(p > \frac{1}{4})$, then $|a_n| = O(n^{2p-1})$.*

THEOREM 16. ([6, Th. 3.3].) *If $f(z) \in X$ is eampv in D with respect to R_0, then $M(r,f) = O(1-r)^{-\alpha}$ implies $|a_n| = O(n^{\alpha-1})$ for all $\alpha > \frac{3}{2}$.*

COROLLARY 17. *If $f(z) \in X$ is eam1v in D with respect to R_0, then $|a_n| = O(n)$.*

THEOREM 18. ([6, Th. 3.8].) *If*

$$f(z) = \sum_{j=0}^{N-1} a_j z^j + a_N z^N + a_{N+k} z^{N+k} + a_{N+2k} z^{N+2k} + \cdots$$

is eampv in D with respect to R_0 and $A(r,R_0) = O(1-r)^{-\gamma}$, $0 \leqq \gamma \leqq 2p/k - \frac{1}{2}$, $1 \leqq k \leqq 4p$, then $|a_n| = O(n^{2p/k-1})$.

COROLLARY 19. *If*

$$f(z) = \sum_{j=0}^{N-1} a_j z^j + \sum_{j=0}^{\infty} a_{N+jk} z^{N+jk}$$

is eampv in D with respect to R_0 and if $f \in X$, then $|a_n| < O(n^{2p/k-1})$ for $1 \leqq k \leqq \frac{4}{3}p$.

THEOREM 20. ([6, Th. 3.9].) *Let $f(z) = \sum a_n z^n$ be eampv in D with respect to R_0. Suppose $a_n = 0$ whenever $n = bm + c$ where b and c are fixed positive integers and m goes from one to infinity. If $A(r,R_0) = O(1-r)^{-\gamma}$, $0 \leqq \gamma \leqq p - \frac{1}{2}$, then $|a_n| = O(n^{p-1})$.*

In particular if $f(z)$ is *eam1v* with coefficients that vanish on an arithmetic sequence, then even if $A(r,R_0)$ grows as $O(1-r)^{-\frac{1}{2}}$, the coefficients of $f(z)$ are still bounded.

COROLLARY 21. *If $f(z) \in X$ satisfies the conditions of Theorem 20, then* $|a_n| = O(n^{p-1})$ *for all* $p \geqq 3/2$.

THEOREM 22. ([5, p. 190].) *Let $f(z) = \sum a_n z^n$ be eampv in D with respect to R_0. Let b and c be positive integers with $1 \leqq c \leqq b$. Suppose there is a positive integer N such that for all integers $n = bm + c$, with m an integer greater than or equal to N, we have $|a_n| \leqq C n^\alpha$ where $\alpha \geqq \max\{p-1, 0\}$. If $A(r,R) = O(1-r)^{-\gamma}$, $0 \leqq \gamma \leqq \alpha + \frac{1}{2}$, then $|a_n| \leqq A(C,p,b,c,f,R_0)\, n^\alpha$ for all n.*

COROLLARY 23. *Suppose $f(z) \in X$ is eampv in D with respect to R_0. Let b and c be positive integers with $1 \leqq c \leqq b$. If there is a positive integer N such that for all integers $n = bm + c$, with m an integer greater than or equal to N, we have $|a_n| = O(n^\alpha)$ where $\alpha \geqq \max(p-1, \frac{1}{2})$, then $|a_n| = O(n^\alpha)$ for all n.*

We make three concluding remarks. First, Spencer's *eampv* functions (*eampv* and locally finite area) trivially satisfy $A(r, R_0) \leqq C(1-r)^{-\gamma}$ with $\gamma = 0$. Therefore the previous theorems hold immediately for such a class of functions.

Second, since for any *eampv* function of locally finite area there is a complex number c such that $g(z) = f(z) + c$ is (globally) *amqv* for some finite q, we see that all of the theorems of Pommerenke's paper [9] can be generalized from the class $\mathfrak{A}$ (the class of all *amqv* functions) to $\mathfrak{A}^*$ (the class of all *eampv* functions of locally finite area). We also take this opportunity to point out that one of his theorems [9, Th. 3] still holds true under the much weaker condition that $f(z)$ have only weak mean p-valence [10, p. 201]. Here is a statement of [9, Th. 3]. If $f(z) = a_0 + \sum_{n=1}^{\infty} a_{n_k} z^{n_k}$ satisfies $\sum_{n=1}^{\infty} 1/n_k < \infty$ and $f(z)$ is *amqv*, then $\sum_{n=1}^{\infty} |a_{n_k}| < \infty$.

Third and finally, we note that the restriction $\alpha \geqq \max(p-1, 0)$ cannot be removed from Theorem 11 or Theorem 22 for any $p \geqq 1$. That is, for any $p \geqq 1$ and any α satisfying $-\infty < \alpha < \max(p-1, 0)$, there is an *eampv* function which does satisfy $|a_n| \leqq C n^\alpha$ on some sequence $n = bm + c$, but for which $M(r,f) \neq O(1-r)^{-(\alpha+1)}$ and also for which $|a_n| \neq O(n^\alpha)$. Consider the functions $f(z) = (1+z^2)^{-p} = 1 + a_2 z^2 + a_4 z^6 + \cdots$ and $f(z)^{\frac{1}{2}} = g(w) = (1+w)^{-p} = 1 + a_2 w + a_4 w^2 + a_6 w^3 + \cdots$. A direct computation shows $a_{2n} \sim n^{p-1}/\Gamma(p)$, $a_{2n+1} = 0$. Hence for any α less than $p-1$, we have $|a_n| \leqq 1 \cdot n^\alpha$ for all even n. But, as is evident, $M(r,f)$ is not $O(1-r)^{-(\alpha+1)}$ nor is $|a_n| = O(n^\alpha)$. It is an open question whether the restriction is necessary if $p < 1$.

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